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A general form of Krylov–Bogoliubov–Mitropolskii method for solving nonlinear partial differential equations

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Abstract

A general asymptotic solution can be obtained for a class of partial differential equations with small nonlinearities whose dominant linear part involves an nth order, $n = 2, 3, \ldots$, time derivative. The method used is an extension of the Krylov–Bogoliubov–Mitropolskii (KBM) method. The formulation as well as the determination of the solution is quite easy. Many authors have extended the KBM method to investigate some physical and mechanical oscillating systems, modelled by either second-order hyperbolic type partial differential equations or certain partial differential equations with third-order time derivative. They mainly extended the method to investigate individual problems. On the contrary, the proposed solution covers various types of nonlinear problems modelled by partial differential equations whose linear part involves second-, third-, etc. order time derivative. Substituting $n = 2, 3$ into the general formula, it can be shown that the formula readily becomes to those extended by several authors. The method is illustrated with a physical problem whose linear part involves a third-order time derivative.

1. Introduction

The Krylov–Bogoliubov–Mitropolskii (KBM) method [1,2] was extended and used by Mitropolskii and Moseenkov [3], Fodchuk [4] and Bojadziev and Lardner [5] for solving second-order hyperbolic type partial differential equations with small nonlinearities. The method

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was developed earlier by Krylov and Bogoliubov [1] for obtaining periodic solutions of second-order ordinary differential equations. Then the method was amplified and justified by Bogoliubov and Mitropolskii [2] and extended by Popov [6] to damped oscillatory processes. Bojadziev and Lardner [7,8] further extended the method to hyperbolic type partial differential equations with time delay and damping. Osiniskii [9,10] investigated partial differential equations whose linear parts involve third-order time derivative. However, Osiniskii did not utilize the KBM method properly. Lardner and Bojadziev [11] observed that Osiniskii's solution gives incorrect results in certain cases. They [11] then provided a solution to the problem considered by Osiniskii [9,10]. Lardner and Bojadziev [11] developed the method using the concept of 'mode coupling' introduced by Devy and Ames [12], who first successfully investigated nonlinear partial differential equations with third-order time derivatives by perturbation technique. They [12] used a two-variable expansion procedure [13] instead of the KBM method. It is clear that a lot of formulae exist to tackle various types of nonlinear problems modelled by partial differential equations. These formulae are mainly derived depending on the order of the differential equation, the damping force, or the nature of the coefficients (either constant coefficients or slowly varying coefficients with time). However, generalizing all these situations a formula can be found. It should be noted that Shamsul [14–16] has presented such general formula for the nonlinear ordinary differential equations. In one [14], a general solution of an nth order ordinary differential equation (with constant coefficients) is given and later this method is extended and applied to similar nonlinear equations in which the coefficients slowly vary with time [15]. In both papers[14,15], the general solutions are considered in terms of some unusual variables rather than amplitudes and phases. Therefore, these solutions need suitable variable transformations to bring them to the formal forms. Recently this problem is solved [16]. Both formulae presented in Refs. [14,15] have been transformed to formal forms which are used directly to determine first-order solutions of second, third, fourth order, etc. equations. Thus the method presented in Ref. [16] is generalized and straightforward. The aim of the present paper is to extend that method to partial differential equations whose linear part contains an nth order time derivative. The general formula is identical to some existing formulae [3–5,7,8] when $n=2$. But the formula reduces to an equivalent form of Lardner and Bojadziev's [11] formula when $n=3$. It should be noted that the reduced formula (the concern of this paper) is much simpler than that of Ref. [11].

2. Method

Considering the following nonlinear partial differential equation:

$$(D^n + c_1 D^{n-1} + \cdots + c_{n-1} D)u - (d_2 D^{n-2} + d_3 D^{n-3} + \cdots + d_n) u_{xx} = \varepsilon F(x, u, u_x, u_t, \ldots), \quad (1)$$

where $D \equiv \partial/\partial t$, subscripts denote partial differentiation with respect to x and t, c_j, $j = 1, 2, \ldots, n-1$; d_j, $j = 2, \ldots, n-2$ are constants, ε is a small parameter and F is a given nonlinear function. In addition to Eq. (1), $u(x,t)$ is required to satisfy a pair of homogeneous boundary conditions involving u and its derivatives at $x=0$ and l:

$$B_j(u) = \beta_{j1} u(0,t) + \beta_{j2} u_x(0,t) + \beta_{j3} u(l,t) + \beta_{j4} u_x(l,t), \quad j = 1, 2, \quad (2)$$

where $\beta_{j,r}$, $r = 1, 2, 3, 4$ are eight constants.

Setting $\varepsilon = 0$ in Eq. (1), the generating equation which possesses the separable solutions is

$$u^{[s]}(x,t,0) = \Phi_s(x)\sum_{j=1}^{n} a_{j,0}^{(s)}\,\mathrm{e}^{\lambda_j^{(s)}t}, \quad s = 1,2,\ldots, \tag{3}$$

where $a_{j,0}^{(s)}$, $j = 1,2,\ldots,n$ are arbitrary constants, $\Phi_s(x)$ satisfies the ordinary differential equation

$$\Phi_s''(x) + \mu_s^2\Phi_s(x) = 0, \quad B_j(\Phi_s) = 0, \quad j = 1,2 \tag{4}$$

and for each values of μ_s, $\lambda_j^{(s)}$, $j = 1,2,\ldots,n$ satisfy the algebraic equation

$$p^n + c_1p^{n-1} + (c_2 + \mu_s^2d_2)p^{n-2} + \cdots + \mu_s^2d_n \equiv \prod_{j=1}^{n}(p - \lambda_j^{(s)}) = 0. \tag{5}$$

The boundary conditions on u holding at $x = 0$ and l translate into corresponding boundary conditions on $\Phi_s(x)$, which combined with Eq. (4), enable the allowed set of eigenfunctions $\{\Phi_s(x)\}$ and eigenvalues $\{\mu_s^2\}$ to be determined. Provided the boundary conditions satisfy the usual conditions for self-adjointness, the set of eigenfunctions form a complete and orthogonal set. By suitable normalization, one gets

$$\int_0^l \Phi_r(x)\Phi_s(x)\,\mathrm{d}x = \delta_{rs}, \tag{6}$$

where δ_{rs} is the Kronecker symbol. In order to solve an initial value problem for the generating equation, one would seek the solution in the form of a sum of separable solutions as

$$u(x,t,0) = \sum_{s=1}^{\infty}\left(\Phi_s(x)\sum_{j=1}^{n} a_{j,0}^{(s)}\mathrm{e}^{\lambda_j^{(s)}t}\right). \tag{7}$$

By virtue of the completeness of the set of eigenfunctions, such solution can meet the initial conditions that u, u_t, $u_{tt},\ldots$ are prescribed at $t = 0$, and the orthogonality condition Eq. (6) enables simple expression for the sets of coefficients $\{a_{j,0}^{(s)}\}$.

Considering the nonlinear equation (1), and supposing that one wishes to find the single mode solution $u^{[s]}(x,t,\varepsilon)$ corresponding to the sth mode equation (3) of the generating equation. Such the solution can be defined formally by the requirement that $|u^{[s]}(x,t,\varepsilon) - u^{[s]}(x,t,0)| \to 0$ as $\varepsilon \to 0$, and correspond to single mode solutions of Eq. (1) previously investigated [3–5,7–11] (when $n = 2$ or 3). Following the KBM method [1,2], as generalized for an nth order ordinary differential equation by Shamsul [13] and as developed for partial differential equation by Mitropolskii and Moseenkov [3], an asymptotic solution of Eq. (1) (which corresponds to μ_s) can be chosen in the form

$$u(x,t,\varepsilon) = u^{(0)}(a_1^{(s)}, a_2^{(s)},\ldots,a_n^{(s)},x,t) + \varepsilon u^{(1)}(a_1^{(s)}, a_2^{(s)},\ldots,a_n^{(s)},x,t) + \varepsilon^2\ldots, \tag{8}$$

where $u^{(0)} = \Phi_s(x)\sum_{j=1}^{n} a_j^{(s)}(t)e^{\lambda_j^{(s)}t}$, and each $a_j^{(s)}$ satisfies a first-order differential equation

$$\dot{a}_j^{(s)} = \varepsilon A_j(a_1^{(s)}, a_2^{(s)}, \ldots, a_n^{(s)}, t) + \varepsilon^2 \ldots \,. \tag{9}$$

Differentiating $u(x, t, \varepsilon)$ n-times with respect to x and t, substituting the derivatives of u and u in Eq. (1), it becomes

$$(D^n + c_1 D^{n-1} + \cdots + c_{n-1} D)u^{(0)} - (d_2 D^{n-2} + d_3 D^{n-3} + \cdots + d_n)u_{xx}^{(0)} \\ + \varepsilon[(D^n + c_1 D^{n-1} + \cdots + c_{n-1} D)u^{(1)} - (d_2 D^{n-2} + d_3 D^{n-3} + \cdots + d_n)u_{xx}^{(1)}] + \varepsilon^2 \cdots = \varepsilon F. \tag{10}$$

It is customary in KBM method that $u^{(1)}$ is expanded in a series of $\Phi_r(x)$, $r = 1, 2, \ldots$ (see Refs. [3–5,7–11] for details), as

$$u^{(1)} = \sum_{r'=1}^{\infty} U^{(r')}(a_1^{(s)}, a_2^{(s)}, \ldots, a_n^{(s)}, t)\,\Phi_{r'}(x). \tag{11}$$

Substituting the values of $u^{(0)}$; $u^{(1)}$ from Eq. (11) into Eq. (10) and utilizing Eq. (4):

$$\Phi_s(\bar{D}^n + c_1\bar{D}^{n-1} + (c_2 + \mu_s^2 d_2)\bar{D}^{n-2} \cdots + \mu_s^2 d_n) \sum_{j=1}^{n} a_j^{(s)}(t)e^{\lambda_j^{(s)}t} \\ + \varepsilon \sum_{r'=1}^{\infty} \Phi_{r'}(\bar{D}^n + c_1\bar{D}^{n-1} + (c_2 + \mu_{r'}^2 d_2)\bar{D}^{n-2} \cdots + \mu_{r'}^2 d_n)\, U^{(r')} + \varepsilon^2 \cdots = \varepsilon F, \tag{12}$$

where $\bar{D} \equiv \mathrm{d}/\mathrm{d}t$.

According to the characteristic Eq. (5), Eq. (12) can be written in a factorized form in $\bar{D}$, as

$$\Phi_s \prod_{k=1}^{n} (\bar{D} - \lambda_j^{(s)}) \sum_{j=1}^{n} a_j^{(s)}(t)e^{\lambda_j^{(s)}t} + \varepsilon \sum_{r'=1}^{\infty} \left(\Phi_{r'} \prod_{k=1}^{n} (\bar{D} - \lambda_j^{(r')}) U^{(r')} \right) + \varepsilon^2 \cdots = \varepsilon F. \tag{13}$$

Now multiplying both sides of Eq. (13) by $\Phi_r(x)$, integrating from 0 to l and making use of Eq. (6):

$$\delta_{rs} \prod_{k=1}^{n} (\bar{D} - \lambda_j^{(s)}) \sum_{j=1}^{n} a_j^{(s)}(t)e^{\lambda_j^{(s)}t} + \varepsilon \prod_{k=1}^{n} (\bar{D} - \lambda_j^{(r)})\, U^{(r)} + \varepsilon^2 \cdots = \varepsilon F^{(r)}(a_1^{(s)}, a_2^{(s)}, \ldots, a_n^{(s)}, t), \tag{14}$$

where $F^{(r)} = \int_0^l F\, \Phi_r(x)\,\mathrm{d}x$. The coefficient with δ_{rs} of Eq. (14) can be rewritten as

$$\sum_{j=1}^{n} \prod_{k=1}^{n} (\bar{D} - \lambda_k^{(s)})(a_j^{(s)}(t)e^{\lambda_j^{(s)}t}) \\ = \sum_{j=1}^{n} \prod_{k=1, k\neq j}^{n} (\bar{D} - \lambda_k^{(s)})((\bar{D} - \lambda_j^{(s)})(a_j^{(s)}(t)e^{\lambda_j^{(s)}t})). \tag{15}$$

Using Eq. (9), one obtains

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - \lambda_j^{(s)} \right)(a_j^{(s)}(t)e^{\lambda_j^{(s)}t}) = \varepsilon A_j e^{\lambda_j^{(s)}t} + O(\varepsilon^2). \tag{16}$$

Therefore,

$$\begin{aligned}\prod_{k=1,\ k\neq j}^{n}\left(\frac{\mathrm{d}}{\mathrm{d}t}-\lambda_k^{(s)}\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}-\lambda_j^{(s)}\right)(a_j^{(s)}(t)\mathrm{e}^{\lambda_j^{(s)}t})\\ =\varepsilon\prod_{k=1,\ k\neq j}^{n}\left(\frac{\mathrm{d}}{\mathrm{d}t}-\lambda_k^{(s)}\right)(A_j\mathrm{e}^{\lambda_j^{(s)}t})+O(\varepsilon^2)\\ =\varepsilon\prod_{k=1,\ k\neq j}^{n}\left(\frac{\partial}{\partial t}-\lambda_k^{(s)}\right)(A_j\mathrm{e}^{\lambda_j^{(s)}t})+O(\varepsilon^2)\end{aligned} \tag{17}$$

On the other hand, the term with ε of Eq. (14) becomes

$$\prod_{k=1}^{n}\left(\frac{\mathrm{d}}{\mathrm{d}t}-\lambda_k^{(r)}\right)\ U^{(r)}=\prod_{k=1}^{n}\left(\frac{\partial}{\partial t}-\lambda_k^{(r)}\right)U^{(r)}+O(\varepsilon). \tag{18}$$

Substituting the results of Eqs. (17) and (18) into Eq. (14) and comparing the coefficients of ε, one obtains

$$\delta_{rs}\left(\sum_{j=1}^{n}\left(\prod_{k=1,\,k\neq j}^{n}(D-\lambda_k^{(s)})(\mathrm{e}^{\lambda_j^{(s)}t}A_j)\right)\right)+\prod_{j=1}^{n}(D-\lambda_j^{(r)})\,U^{(r)}=F^{(r)}(a_1^{(s)},a_2^{(s)},\ldots,a_n^{(s)},t). \tag{19}$$

According to the assumptions followed in Ref. [14], it is easy to determine $A_j,\ j=1,2,\ldots,n$ and $U^{(r)},\ r=1,2,\ldots$ from Eq. (19). To do this it is restricted that $U^{(s)}$ excludes fundamental terms which are included in the series expansion equation (8) at order ε^0 [3–5,7,8,11,12]. But the formula Eq. (19) is not in a usual form and the determination of the solution is not straightforward. First, $A_j,\ j=1,2,\ldots,n$ and $U^{(r)},\ r=1,2,\ldots$ are to be determined from Eq. (19). Then all these functional values are substituted into Eqs. (8) and (9) and some suitable variable transformations are used to obtain the formal solution from Eqs. (8) and (9). However, under the same variable transformations, Eq. (19) can be brought to a usual form, i.e., in terms of amplitude and phase variables. For both even and odd values of n, the general formula is useful to determine all oscillatory modes. When n is an odd number, there exists a nonoscillatory mode (purely exponential type, see Ref. [11]) which can be found by solving an additional equation (see Ref. [16] for details). First, considering the situation when n is an even number, i.e., $n=2l,\ l=1,2,\ldots$. In this case Eq. (19) can be written as

$$\begin{aligned}\delta_{rs}\sum_{l=1}^{n/2}\left(\prod_{k=1,\ k\neq 2l-1,\ 2l}^{n}((D-\lambda_{2l}^{(s)})\ (\mathrm{e}^{\lambda_{2l-1}^{(s)}t}A_{2l-1})+(D-\lambda_{2l-1}^{(s)})\ (\mathrm{e}^{\lambda_{2l}^{(s)}t}A_{2l}))\right)\\ +\prod_{j=1}^{n}(D-\lambda_j^{(r)})U^{(r)}=F^{(r)}(a_1^{(s)},a_2^{(s)},\ldots,a_n^{(s)},t).\end{aligned} \tag{20}$$

Now with change of variables $a_j^{(s)},\ j=1,2,\ldots,n$ by $a_{2l-1}^{(s)}=\frac{1}{2}\alpha_l\mathrm{e}^{\mathrm{i}\varphi_l},\ a_{2l}^{(s)}=\frac{1}{2}\alpha_l\mathrm{e}^{-\mathrm{i}\varphi_l}$ ($\alpha_l,\ \varphi_l$ are, respectively, amplitude and phase variables) together with substitutions $\lambda_{2l-1}^{(s)}=-\mu_l+\mathrm{i}\omega_l,\ \lambda_{2l}^{(s)}=$

$-\mu_l - \mathrm{i}\omega_l$, $A_{2l-1} = \frac{1}{2}(\tilde{A}_l - \mathrm{i}\alpha_l\tilde{B}_l)\mathrm{e}^{\mathrm{i}\varphi_l}$, $A_{2l} = \frac{1}{2}(\tilde{A}_l - \mathrm{i}\alpha_l\tilde{B}_l)\mathrm{e}^{-\mathrm{i}\varphi_l}$, Eq. (20) becomes

$$\delta_{rs}\sum_{l=1}^{n/2}\left(\prod_{k=1,\ k\neq 2l-1,\ 2l}^{n}(D-\lambda_j^{(s)})\ (\mathrm{e}^{-\mu_l t}\{(D\tilde{A}_l - 2\omega_l\alpha_l\tilde{B}_l)\cos\psi_l - (2\omega_l\tilde{A}_l + \alpha_l D\tilde{B}_l)\sin\psi_l\})\right)$$
$$+\prod_{j=1}^{n}(D-\lambda_j^{(r)})\ U^{(r)} = F^{(r)}(\alpha_1,\alpha_2,\ldots,\alpha_{n/2},\psi_1,\psi_2,\ldots,\psi_{n/2},t),\quad \psi_l = \omega_l t + \varphi_l. \tag{21}$$

It can be shown that Eq. (21) is similar to Bojadziev and Lardner's [8] formula which was derived for obtaining the first-order solution of $u_{tt} + c_1u_t - d_2u_{xx} = \varepsilon F(x,u,u_x,u_t)$. For $n = 2$, Eq. (21) readily becomes

$$\delta_{rs}\mathrm{e}^{-\mu_1 t}\{\cos\psi_1(D\tilde{A}_1 - 2\omega_1\alpha_1\tilde{B}_1) - \sin\psi_1(2\omega_1\tilde{A}_1 + \alpha_1 D\tilde{B}_1)\}$$
$$+ (D^2 + 2\mu_1 D + \mu_1^2 + \omega_1^2)U^{(r)} = F^{(r)}(\alpha_1,\psi_1,t),\quad \psi_1 = \omega_1 t + \varphi_1. \tag{22}$$

It is obvious that Eq. (22) is in an equivalent form of Bojadziev and Lardner's formula [8] (see the article for details). Eq. (22) reduces to Bojadziev and Lardner's [5] formula when $\mu_1 \to 0$ or $\mu_1 = O(\varepsilon)$. Therefore, when $n = 2$ and $\mu_1 = 0$, Eq. (21) is identical to Bojadziev and Lardner's [5] formula. Bojadziev and Lardner's [5,8] solved Eq. (22) for the unknown functions $\tilde{A}_1$, $\tilde{B}_1$ and $U^{(r)}$, subject to the conditions that $U^{(s)}$ excludes first harmonic terms, $\cos\psi_1$ and $\sin\psi_1$. Following this assumption, one can solve Eq. (21) for all even values of n. When n is an odd number, Eq. (19) can be written as

$$\delta_{rs}\sum_{l=1}^{(n-1)/2}\left(\prod_{k=1,k\neq 2l-1,2l}^{n}((D-\lambda_{2l}^{(s)})(\mathrm{e}^{\lambda_{2l-1}^{(s)}t}A_{2l-1}) + (D-\lambda_{2l-1}^{(s)})(\mathrm{e}^{\lambda_{2l}^{(s)}t}A_{2l}))\right)$$
$$+\prod_{k=1}^{n-1}(D-\lambda_k^{(s)})(\mathrm{e}^{\lambda_n^{(s)}t}A_n) + \prod_{j=1}^{n}(D-\lambda_j^{(s)})u^{(1)} = F^{(r)}. \tag{23}$$

Therefore, Eq. (21) is still valid when n is an odd number, but the functions $\tilde{A}_l$, $\tilde{B}_l$ exist for $l = 1, 2, \ldots, (n-1)/2$, while the unknown function A_n would be determined from the additional equation

$$\prod_{k=1}^{n-1}(D-\lambda_k^{(s)})\,(\mathrm{e}^{\lambda_n^{(s)}t}A_n) = \text{all term of } \mathrm{e}^{\lambda_n t} \text{ of the expansion of } F^{(s)}. \tag{24}$$

Therefore, the determination of first order solution of Eq. (1) is clear whether n is an even or an odd number. The method can be carried out to higher order approximations in a similar way. However, owing to the rapidly growing algebraic complexity for the derivation of the formulae, the solution is in general confined to a low order, usually the first [14–16].

3. Example

3.1. Vibration of a viscoelastic rod in where a third-order time derivative is present

As an example of the above procedure, consider a nonlinear mechanical elastic system with internal friction and relaxation, namely the longitudinal vibrations of a viscoelastic rod or torsional vibrations of a cylinder. Let x be a coordinate along the rod and let $u(x,t)$, $e(x,t)$ and $\sigma(x,t)$ denote, respectively, the longitudinal displacement, axial strain and axial stress at time t at the particle whose positional coordinate is x in the unstressed state of the rod. In the case of torsional vibrations, e and σ denote tangential strain and stress. The strain–displacement relation is

$$e = u_x \tag{25}$$

and the equation of motion is

$$\rho u_{tt} = \sigma_x, \tag{26}$$

where ρ is the density of the rod.

The constitutive equation of the material, proposed and used by Osiniski [9,10] is supposed to be of the form (see also Refs. [11,12,17])

$$\sigma + \beta\sigma_t + \beta_1\sigma_t^3 = Ke + K_1e^3 + \Gamma e_t + \Gamma_1 e_t^3, \tag{27}$$

where the nonlinear terms are small compared to the linear ones. The terms with coefficients K and K_1 represent, respectively, the linear and nonlinear elasticity, the terms with coefficients Γ and Γ_1 correspond, respectively, to linear and nonlinear viscous damping, and the terms with coefficients β and β_1 reflect linear and nonlinear relaxation. In some particular cases it is considered that $\Gamma_1 = \beta_1 = 0$ (see Refs. [9–11,17] for details). Now eliminating e and σ from Eqs. (25) to (27), the system takes the form of Eq. (1) as

$$u_{ttt} + \beta^{-1}u_{tt} - (\Gamma\beta^{-1}\rho^{-1}u_{xxt} + K\beta^{-1}\rho^{-1}u_{xx}) = 3K_1\beta^{-1}\rho^{-1}u_x^2u_{xx}. \tag{28}$$

Here $n=3$, $c_1 = \beta^{-1}$, $c_2 = 0$, $d_2 = \Gamma\beta^{-1}\rho^{-1}$, $d_3 = K\beta^{-1}\rho^{-1}$ and $\varepsilon = 3K_1\beta^{-1}\rho^{-1}$, $F = u_x^2u_{xx}$. Here $j = 1,2,3$ and $l_1 = 1$ only. Let us consider $\lambda_1^{(s)} = -\zeta_s + \mathrm{i}\omega_s$, $\lambda_2^{(s)} = -\zeta_s - \mathrm{i}\omega_s$ $\lambda_3^{(s)} = -\xi_s$, $\alpha_1^{(s)} = b$, $a_3^{(s)} = a$ and $\psi_1^{(s)} = \psi$; so that Eq. (21) becomes

$$\begin{aligned}&\delta_{rs}((D+\xi_s)[\mathrm{e}^{-\zeta_s t}\{(D\tilde{A}_1 - 2\omega_s b\tilde{B}_1)\cos\psi - (2\omega_s\tilde{A}_1 + bD\tilde{B}_1)\sin\psi\}]\\ &\quad + \{(D+\xi_s)^2 + \omega_s^2\}(A_3\mathrm{e}^{-\xi_s t})) + (D+\xi_r)[(D+\xi_r)^2 + \omega_r^2]\,U^{(r)} = F^{(r)}(b\mathrm{e}^{-\zeta_s t}\cos\psi + a\mathrm{e}^{-\xi_s t})^3,\end{aligned} \tag{29}$$

where $F^{(r)} = \int_0^l \Phi_s'^2\Phi_s''\Phi_r\,\mathrm{d}x$. When $r = s$, Eq. (29) can be separated into the three equations as (subject to the conditions that $U^{(s)}$ excludes the first harmonic terms and the terms of $a\mathrm{e}^{-\xi_s t}$)

$$\begin{aligned}&(D+\xi_s)[\mathrm{e}^{-\zeta_s t}\{(D\tilde{A}_1 - 2\omega_s b\tilde{B}_1)\cos\psi - (2\omega_s\tilde{A}_1 + bD\tilde{B}_1)\sin\psi\}]\\ &\quad = F^{(s)}(\tfrac{3}{4}b^3\mathrm{e}^{-3\zeta_s t} + 3ba^2\mathrm{e}^{-(\zeta_s+2\xi_s)t})\cos\psi,\end{aligned} \tag{30}$$

$$\{(D+\xi_s)^2 + \omega_s^2\}(A_3\mathrm{e}^{-\xi_s t}) = F^{(s)}(\tfrac{3}{2}b^2a\mathrm{e}^{-(2\zeta_s+\xi_s)t} + a^3\mathrm{e}^{-3\xi_s t}) \tag{31}$$

and

$$(D+\xi_s)[(D+\xi_s)^2+\omega_s^2]U^{(s)} = F^{(s)}(\tfrac{3}{2}b^2 a \mathrm{e}^{-(2\zeta_s+\xi_s)t}\cos 2\psi + \tfrac{1}{4}b^3\mathrm{e}^{-3\zeta_s t}\cos 3\psi). \tag{32}$$

When $r \neq s$, Eq. (29) becomes to a simple form

$$(D+\xi_r)[(D+\xi_r)^2+\omega_r^2]\,U^{(r)} = F^{(r)}(b\mathrm{e}^{-\zeta_s t}\cos\psi + a\mathrm{e}^{-\xi_s t})^3. \tag{33}$$

After integration once, Eq. (30) becomes

$$\begin{aligned}&\mathrm{e}^{-\zeta_s t}\{\cos\psi(D\tilde{A}_1 - 2\omega_s b\tilde{B}_1) - \sin\psi(2\omega_s\tilde{A}_1 + bD\tilde{B}_1)\}\\ &\quad = \tfrac{3}{4}F^{(s)}b^3\mathrm{e}^{-3\zeta_s t}\times\frac{(\xi_s-3\zeta_s)\cos\psi+\omega_s\sin\psi}{(\xi_s-3\zeta_s)^2+\omega_s^2}\\ &\qquad - 3F^{(s)}a^2 b\mathrm{e}^{-(2\xi_s+\zeta_s)t}\times\frac{(\xi_s+\zeta_s)\cos\psi-\omega_s\sin\psi}{(\xi_s+\zeta_s)^2+\omega_s^2}.\end{aligned} \tag{34}$$

Equating the coefficients $\cos\psi$ and $\sin\psi$ on both sides of Eq. (34), one obtains

$$\begin{aligned}\mathrm{e}^{-\zeta_s t}(D\tilde{A}_1 - 2\omega_s b\tilde{B}_1) &= \tfrac{3}{4}F^{(s)}b^3\mathrm{e}^{-3\zeta_s t}\times\frac{(\xi_s-3\zeta_s)}{(\xi_s-3\zeta_s)^2+\omega_s^2} - 3F^{(s)}a^2 b\mathrm{e}^{-(2\xi_s+\zeta_s)t}\times\frac{(\xi_s+\zeta_s)}{(\xi_s+\zeta_s)^2+\omega_s^2},\\ -\mathrm{e}^{-\zeta_s t}(2\omega_s\tilde{A}_1 + bD\tilde{B}_1) &= \tfrac{3}{4}F^{(s)}b^3\mathrm{e}^{-3\zeta_s t}\times\frac{\omega_s}{(\xi_s-3\zeta_s)^2+\omega_s^2} + 3F^{(s)}a^2 b\mathrm{e}^{-(2\xi_s+\zeta_s)t}\times\frac{\omega_s}{(\xi_s+\zeta_s)^2+\omega_s^2}.\end{aligned} \tag{35}$$

It is obvious that the solution of Eq. (35) takes the form

$$\tilde{A}_1 = m_2 b^3\mathrm{e}^{-2\zeta_s t} + m_1 a^2 b\mathrm{e}^{-2\xi_s t}, \qquad \tilde{B}_1 = n_2 b^2\mathrm{e}^{-2\zeta_s t} + n_1 a^2\mathrm{e}^{-2\xi_s t}, \tag{36}$$

where the unknown coefficients $m_2,\ n_2,\ m_1,\ n_1$ satisfy the algebraic equations

$$\begin{aligned}&-2\zeta_s m_2 - 2\omega_s n_2 = \frac{\tfrac{3}{4}F^{(s)}(\xi_s-3\zeta_s)}{(\xi_s-3\zeta_s)^2+\omega_s^2}, \quad 2\omega_s m_2 - 2\zeta_s n_2 = \frac{\tfrac{3}{4}F^{(s)}\omega_s}{(\xi_s-3\zeta_s)^2+\omega_s^2},\\ &-2\zeta_s m_1 - 2\omega_s n_1 = \frac{-3F^{(s)}(\xi_s+\zeta_s)}{(\xi_s+\zeta_s)^2+\omega_s^2}, \quad 2\omega_s m_1 - 2\zeta_s n_1 = \frac{-3F^{(s)}\omega_s}{(\xi_s+\zeta_s)^2+\omega_s^2}.\end{aligned} \tag{37}$$

The solution of Eq. (37) is

$$\begin{aligned}m_2 &= \frac{-3F^{(s)}[\zeta_s(\xi_s-3\zeta_s)+\omega_s^2]}{8(\zeta_s^2+\omega_s^2)[(\xi_s-3\zeta_s)^2+\omega_s^2]}, \quad n_2 = \frac{3F^{(s)}\omega_s(-\xi_s+4\zeta_s)}{8(\zeta_s^2+\omega_s^2)[(\xi_s-3\zeta_s)^2+\omega_s^2]},\\ m_1 &= \frac{3F^{(s)}(\xi_s^2+\xi_s\zeta_s-\omega_s^2)}{2(\xi_s^2+\omega_s^2)[(\xi_s+\zeta_s)^2+\omega_s^2]}, \quad n_1 = \frac{3F^{(s)}\omega_s(2\xi_s+\zeta_s)}{2(\xi_s^2+\omega_s^2)[(\xi_s+\zeta_s)^2+\omega_s^2]}.\end{aligned} \tag{38}$$

Eq. (31) can be solved easily. The solution becomes

$$A_3 = l_2 ab^2\mathrm{e}^{-2\zeta t} + l_1 a^3\mathrm{e}^{-2\xi t}, \tag{39}$$

where

$$l_2 = \frac{3F^{(s)}}{2[(\xi_s + \zeta_s)^2 + \omega_s^2]}, \quad l_1 = \frac{F^{(s)}}{(3\xi_s - \zeta_s)^2 + \omega_s^2}. \tag{40}$$

Solving Eqs. (32) and (33), one obtains

$$\begin{aligned} U^{(r)} = F^{(r)}\Phi_r\{&b^3 e^{-3\zeta t}((C_1^{(r)}\cos\psi + S_1^{(r)}\sin\psi) + (C_3^{(r)}\cos 3\psi + S_3^{(r)}\sin 3\psi)) \\ &+ 3b^2 a e^{-(2\zeta+\xi)t} \times (C_0^{(r)} + (C_2^{(r)}\cos 2\psi + S_2^{(r)}\sin 2\psi)) \\ &+ 3ba^2 e^{-(\zeta+2\xi)t}(\bar{C}_1^{(r)}\cos\psi + \bar{S}_1^{(r)}\sin\psi) + \bar{C}_0^{(r)} a^3 e^{-3\xi t}\} \end{aligned} \tag{41}$$

together with $C_1^{(s)} = D_1^{(s)} = \bar{C}_1^{(s)} = \bar{D}_1^{(s)} = C_0^{(s)} = \bar{C}_0^{(s)} = 0$ and

$$C_1^{(r)} = \frac{(\xi_r - 3\zeta_s)(\zeta_r - 3\zeta_s)^2 + (\xi_r - 3\zeta_s)\omega_r^2 - (\xi_r + 2\zeta_r - 9\zeta_s)\omega_s^2}{[(\xi_r - 3\zeta_s)^2 + \omega_s^2][(\xi_r - 3\zeta_s)^2 + (\omega_r + \omega_s)^2][(\xi_r - 3\zeta_s)^2 + (\omega_r - \omega_s)^2]},$$

$$S_1^{(r)} = \frac{((\xi_r - 3\zeta_s)(2\xi_r + \zeta_r - 9\zeta_s) + \omega_r^2 - \omega_s^2)\omega_s}{[(\xi_r - 3\zeta_s)^2 + \omega_s^2][(\xi_r - 3\zeta_s)^2 + (\omega_r + \omega_s)^2][(\xi_r - 3\zeta_s)^2 + (\omega_r - \omega_s)^2]},$$

$$C_3^{(r)} = \frac{(\xi_r - 3\zeta_s)\,(\zeta_r - 3\zeta_s)^2 + (\xi_r - 3\zeta_s)\,\omega_r^2 - 9(\xi_r + 2\zeta_r - 9\zeta_s)\,\omega_s^2}{[(\xi_r - 3\zeta_s)^2 + 9\omega_s^2]\,[(\zeta_r - 3\zeta_s)^2 + (\omega_r + 3\omega_s)^2]\,[(\xi_r - 3\zeta_s)^2 + (\omega_r - 3\omega_s)^2]},$$

$$S_3^{(r)} = \frac{3((\zeta_r - 3\zeta_s)\,(2\xi_r + \zeta_r - 9\zeta_s) + \omega_r^2 - 9\omega_s^2)\,\omega_s}{[(\xi_r - 3\zeta_s)^2 + 9\omega_s^2]\,[(\zeta_r - 3\zeta_s)^2 + (\omega_r + 3\omega_s)^2]\,[(\xi_r - 3\zeta_s)^2 + (\omega_r - 3\omega_s)^2]},$$

$$\bar{C}_1^{(r)} = \frac{(\xi_r - 2\xi_s - \zeta_s)\,(\zeta_r - 2\xi_s - \zeta_s)^2 + (\xi_r - 2\xi_s - \zeta_s)\,\omega_r^2 - (\xi_r + 2\zeta_r - 6\xi_s - 3\zeta_s)\,\omega_s^2}{[(\xi_r - 2\xi_s - \zeta_s)^2 + \omega_s^2]\,[(\zeta_r - 2\xi_s - \zeta_s)^2 + (\omega_r + \omega_s)^2]\,[(\zeta_r - 2\xi_s - \zeta_s)^2 + (\omega_r - \omega_s)^2]},$$

$$\bar{S}_1^{(r)} = \frac{((\zeta_r - 2\xi_s - \zeta_s)\,(2\xi_r + \zeta_r - 6\xi_s - 3\zeta_s) + \omega_r^2 - \omega_s^2)\,\omega_s}{[(\xi_r - 2\xi_s - \zeta_s)^2 + \omega_s^2]\,[(\zeta_r - 2\xi_s - \zeta_s)^2 + (\omega_r + \omega_s)^2]\,[(\zeta_r - 2\xi_s - \zeta_s)^2 + (\omega_r - \omega_s)^2]},$$

$$C_2^{(r)} = \frac{(\xi_r - \xi_s - 2\zeta_s)\,(\zeta_r - \xi_s - 2\zeta_s)^2 + (\xi_r - \xi_s - 2\zeta_s)\,\omega_r^2 - 4(\xi_r + 2\zeta_r - 3\xi_s - 6\zeta_s)\,\omega_s^2}{[(\xi_r - \xi_s - 2\zeta_s)^2 + 4\omega_s^2]\,[(\zeta_r - \xi_s - 2\zeta_s)^2 + (\omega_r + 2\omega_s)^2]\,[(\zeta_r - \xi_s - \zeta_s)^2 + (\omega_r - 2\omega_s)^2]},$$

$$S_2^{(r)} = \frac{2((\zeta_r - \xi_s - 2\zeta_s)\,(2\xi_r + \zeta_r - 3\xi_s - 6\zeta_s) + \omega_r^2 - 4\omega_s^2)\,\omega_s}{[(\xi_r - \xi_s - 2\zeta_s)^2 + 4\omega_s^2]\,[(\zeta_r - \xi_s - 2\zeta_s)^2 + (\omega_r + 2\omega_s)^2]\,[(\zeta_r - \xi_s - \zeta_s)^2 + (\omega_r - 2\omega_s)^2]},$$

$$C_0^{(r)} = \frac{1}{(\xi_r - \xi_s - 2\zeta_s)\,[(\zeta_r - \xi_s - 2\zeta_s)^2 + \omega_r^2]}, \quad \bar{C}_0^{(r)} = \frac{1}{(\xi_r - 3\zeta_s)\,[(\zeta_r - 3\zeta_s)^2 + \omega_r^2]}.$$

Thus the first-order approximate solution of Eq. (28) is

$$u(x,t) = \Phi_s(x)(b e^{-\zeta_s t}\cos\psi + a e^{-\xi_s t}) + \varepsilon \sum_{r=1} U^{(r)}, \tag{42}$$

where a, b and ψ are solution of

$$\begin{aligned}\dot{b} &= \varepsilon(m_2 b^3 \mathrm{e}^{-2\zeta_s t} + m_1 a^2 b \mathrm{e}^{-2\xi_s t}),\\ \dot{\psi} &= \omega_s + \varepsilon(n_2 b^2 \mathrm{e}^{-2\zeta_s t} + n_1 a^2 \mathrm{e}^{-2\xi_s t}),\\ \dot{a} &= \varepsilon(l_2 a b^2 \mathrm{e}^{-2\zeta_s t} + l_1 a^3 \mathrm{e}^{-2\xi_s t}),\end{aligned} \tag{43}$$

and $U^{(r)}$ is given by Eq. (41). In general, Eq. (43) is solved by a numerical technique [11,17]. Eq. (43) has an approximate solution when the damping force is significant. In this situation the perturbation solution is fully independent of the numerical method. Bojadziev et al. [11,17] always solved Eq. (43) by Runge–Kutta fourth-order procedure. Shamsul [14] solved Eq. (43) by assuming that a and b are constants in the right-hand sides of Eq. (43).

4. Similar systems with varying coefficients

The method can be used for similar nonlinear problems with varying coefficients, namely

$$(D^n + c_1(\tau)D^{n-1} + \cdots + c_{n-1}(\tau)D)u - (d_2(\tau)D^{n-2} + \cdots + d_n(\tau))u_{xx} = \varepsilon F(\tau, x, u, u_x, u_t, \ldots), \tag{44}$$

where $\tau = \varepsilon t,\ c_j(\tau), d_j(\tau) \geqslant 0$. The coefficients in Eq. (44) are slowly varying in that their time derivatives are proportional to ε [15,16,18].

Setting $\varepsilon = 0,\ \tau = \tau_0 =$Const. in Eq. (44), the unperturbed solution of this equation becomes

$$u^{[s]}(x, t, 0) = \Phi_s(x) \sum_{j=1}^{n} a_{j,0}^{(s)} \mathrm{e}^{\lambda_j^{(s)}(\tau_0)\, t}, \quad s = 1, 2, \ldots, \tag{45}$$

where $\lambda_j^{(s)}(\tau_0),\ j = 1,\ 2,\ \ldots, n$ are constant, but $\lambda_j^{(s)}(\tau)$ vary slowly with time when $\varepsilon \neq 0$.

An asymptotic solution of Eq. (44) can be chosen in the form

$$u(x, t, \varepsilon) = u^{(0)}(a_1^{(s)}, a_2^{(s)}, \ldots, a_n^{(s)}, x, \tau) + \varepsilon\, u^{(1)}(a_1^{(s)}, a_2^{(s)}, \ldots, a_n^{(s)}, x, \tau) + O(\varepsilon^2), \tag{46}$$

where $u^{(0)} = \Phi_s(x)\sum_{j=1}^{n} a_j^{(s)}(t)$, and $a_j^{(s)}$ satisfies a equation

$$\dot{a}_j^{(s)} = \lambda_j^{(s)} a_j^{(s)} + \varepsilon\, A_j(a_1^{(s)}, a_2^{(s)}, \ldots, a_n^{(s)}, \tau) + O(\varepsilon^2). \tag{47}$$

Here the corresponding equation to Eq. (19) becomes (for an even value of n)

$$\begin{aligned}&\delta_{rs}\left(\sum_{j=1}^{n}\left(\prod_{k=1,k\neq j}^{n} (\Omega - \lambda_k^{(s)})A_j\right) + \sum_{j=1}^{n}\left(\sum_{k=1,k\neq j}^{n}\left(\prod_{r=k}^{k+n-3} (\lambda_{2l-1}^{(s)} - \lambda_r^{(s)})\right)\lambda_{2l-1}^{(s)'}\right)\right)\\ &\quad + \prod_{j=1}^{n} (\Omega - \lambda_j^{(s)})U^{(r)} = F^{(r)}(a_1^{(s)}, a_2^{(s)}, \ldots, a_n^{(s)}, \tau),\end{aligned} \tag{48}$$

where

$$\Omega = \sum_{j=1}^{n} \lambda_j^{(s)} a_j^{(s)} \frac{\partial}{\partial a_j^{(s)}}, \quad \lambda_j^{(s)'} = \frac{\mathrm{d}\lambda_j^{(s)}}{\mathrm{d}\tau}, \quad j = 1, 2, \ldots, n.$$

By using transformation $a^{(s)}_{2l-1} = \frac{1}{2} b_l e^{i\psi_l}$, $a^{(s)}_{2l} = \frac{1}{2} b_l e^{-i\psi_l}$ together with substitutions $\lambda^{(s)}_{2l-1} = -\mu^{(s)}_l + i\omega^{(s)}_l$, $\lambda^{(s)}_{2l} = -\mu^{(s)}_l - i\omega^{(s)}_l$, $A_{2l-1} = \frac{1}{2}(\tilde{A}_l + i\tilde{B}_l)$, $A_{2l} = \frac{1}{2}(\tilde{A}_l - i\tilde{B}_l)$, Eq. (48) becomes

$$\delta_{rs} \sum_{l=1}^{n/2} \left(\prod_{k=1,\ k \neq 2l-1,2l}^{n} (\Omega - \lambda^{(s)}_k)[\cos \psi_l \{(\Theta + \mu^{(s)}_l)\tilde{A}_l - (2\omega^{(s)}_l b_l \tilde{B}_l)\} - \sin \psi_l (2\omega^{(s)}_l \tilde{A}_l + b_l \Theta \tilde{B}_l)] \right)$$
$$+ \delta_{rs} \sum_{l=1}^{n/2} (-(\mu^{(s)\prime}_l P^{(s)}_l + \omega^{(s)\prime}_l Q^{(s)}_l) \cos \psi_l + (\mu^{(s)\prime}_l P^{(s)}_l - \omega^{(s)\prime}_l Q^{(s)}_l) \sin \psi_l)$$
$$+ \prod_{j=1}^{n} (\Omega - \lambda^{(s)}_j) U^{(r)} = F^{(r)}(b_1, b_2, \ldots, b_{n/2}, \psi_1, \psi_2, \ldots, \psi_{n/2}), \tag{49}$$

where

$$P^{(s)}_l + iQ^{(s)}_l = \left(\prod_{r=k}^{k+n-3} (\lambda^{(s)}_{2l-1} - \lambda^{(s)}_r) \right) \lambda^{(s)\prime}_{2l-1}, \quad P^{(s)}_l - iQ^{(s)}_l = \left(\prod_{r=k}^{k+n-3} (\lambda^{(s)}_{2l} - \lambda^{(s)}_r) \right) \lambda^{(s)\prime}_{2l}$$

and $\Theta = \sum_{j=1}^{n/2} \mu_j b_j \frac{\partial}{\partial b_j}$. From Eq. (49), it is possible to determine the unknown functions $\tilde{A}_l$, $\tilde{B}_l$ and $U^{(r)}$ in a similar way as determined in Section 3. The method can be used when n is an odd number. Eq. (49) is identical to Eq. (21) when the coefficients of Eq. (44) become constants.

5. General discussion of the results

A general and straightforward formula Eq. (21) is found and used to determine the first approximate solution of the partial differential equation Eq. (1). Eq. (19) can be used to find the same result according to Ref. [14]. But the solution is determined in terms of the unusual variables $a^{(s)}_j, j = 1, 2, \ldots n$. Under the variable transformation $a^{(s)}_{2l-1} = \frac{1}{2}\alpha_l e^{i\varphi_l}$, $a^{(s)}_{2l} = \frac{1}{2}\alpha_l e^{-i\varphi_l}$, $l = 1, 2, \ldots n/2$ or $(n-1)/2$, the solution is brought to the amplitude-phase form. Moreover, substituting $n = 2, 3, \ldots$ in Eq. (21), all the previous formulae (presented by several authors) can be found quickly. On the contrary, a direct attempt to find such a general formula is too difficult. Though two step calculations are needed to determine the formula Eq. (21), the process is very simple and easier than the classical techniques [1–3] even if $n = 2$.

In this paper, a single mode solution is dealt with. However, the method can be extended to a general initial value problem in which the solution is considered as a sum of the modes of the form Eq. (42). But it is a tremendously difficult task to determine such general solutions when $n \geqslant 3$. Sometimes, the lowest order solution of Eq. (1) of the following form:

$$u(x,t) = \sum_{N=1} \Phi_N(x)(b_N e^{-\zeta_N t} \cos \psi_N + a_N e^{-\xi_N t}), \tag{50}$$

is investigated (see Ref. [11]). Here b_N, ψ_N and a_N satisfy the set of first-order differential equations:

$$\dot{b}_N = \varepsilon \tilde{A}_N(b_N, a_N),$$
$$\dot{\psi}_N = \omega_N + \tilde{B}_N(b_N, a_N),$$
$$\dot{a}_N = \tilde{C}_N(b_N, a_N). \quad (51)$$

The method can be extended to similar nonlinear systems in which the coefficients slowly vary with time, i.e., for Eq. (44). The general formula equation (49) is useful in this case. If the coefficients become constant, Eq. (49) readily reduces to Eq. (21).

6. Conclusion

The KBM method [1,2,14–16] is extended and used to obtain asymptotic solutions of the nonlinear partial differential equations whose linear part involves an nth order time derivative. The method is a generalization of asymptotic method [1–3] and the formulae Eq. (21) and Eq. (49) are useful in obtaining the first approximate solution of a partial differential equation with constant and slowly varying coefficients. The formulation as well as the determination of the solution is simpler than all previous formulae which were derived individually to tackle some physical and mechanical problems modelled by the partial differential equation Eq. (1). For different values of n as well as for various damping conditions, the formula is used arbitrarily. Thus it is no longer necessary to treat the individual partial differential equations as well as individual cases (e.g., undamped, damped and strongly damped systems) separately.

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